

# THE CYCLOTOMIC POLYNOMIAL TOPOLOGICALLY

GREGG MUSIKER AND VICTOR REINER

*To the memory of Mark Feshbach.*

ABSTRACT. We interpret the coefficients of the cyclotomic polynomial in terms of simplicial homology.

## 1. INTRODUCTION

This paper studies the cyclotomic polynomial  $\Phi_n(x)$ , which is defined as the minimal polynomial over  $\mathbb{Q}$  for any primitive  $n^{\text{th}}$  root of unity  $\zeta$  in  $\mathbb{C}$ . It is monic, irreducible, and has degree given by the Euler phi function  $\phi(n)$ , with formula

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta^j).$$

The equation

$$(1) \quad x^n - 1 = \prod_{d|n} \Phi_d(x)$$

gives a recurrence showing that all coefficients of  $\Phi_n(x)$  lie in  $\mathbb{Z}$ .

Although well-studied, the coefficients of  $\Phi_n(x)$  are mysterious [2, 10, 11, 15, 16, 18, 31]. We offer here two interpretations for their magnitudes, as orders of cyclic groups. In the initial interpretation (Corollary 5 below) this group is a quotient of the free abelian group  $\mathbb{Z}[\zeta]$  by a certain full rank sublattice.

The second interpretation is topological, given by Theorem 1 below, as the torsion in the homology of a certain simplicial complex associated with a squarefree integer  $n = p_1 \cdots p_d$ . These simplicial complexes originally arose in the work of Bolker [6], reappeared in the work of Kalai [14] and Adin [1] on higher-dimensional matrix-tree theorems, and were shown to be connected with cyclotomic extensions in work of J. Martin and the second author [19]. We review these simplicial complexes briefly here in order to state the result; see Section 4 for more details.

Given a positive integer  $p$ , let  $K_p$  denote a 0-dimensional abstract simplicial complex having  $p$  vertices<sup>1</sup>, which we will label by the residues

$$\{0 \bmod p, 1 \bmod p, \dots, (p-1) \bmod p\}$$

for reasons that will become clear in a moment.

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<sup>1</sup>Note that here  $K_p$  does *not* refer to a complete graph on  $p$  vertices; we hope that this causes no confusion.

Given primes  $p_1, \dots, p_d$ , let

$$K_{p_1, \dots, p_d} := K_{p_1} * \dots * K_{p_d}$$

be the *simplicial join*, [23, §62], of  $K_{p_1}, \dots, K_{p_d}$ . This is a pure  $(d-1)$ -dimensional abstract simplicial complex, that may be thought of as the *complete  $d$ -partite complex* on vertex sets  $K_{p_1}$  through  $K_{p_d}$  of sizes  $p_1, \dots, p_d$ .

The *facets* (maximal simplices) of  $K_{p_1, \dots, p_d}$  are labelled by sequences of residues  $(j_1 \bmod p_1, \dots, j_d \bmod p_d)$ . Denoting the squarefree product  $p_1 \cdots p_d$  by  $n$ , the Chinese Remainder Theorem isomorphism

$$(2) \quad \mathbb{Z}/p_1\mathbb{Z} \times \dots \times \mathbb{Z}/p_d\mathbb{Z} \xrightarrow{\Xi} \mathbb{Z}/n\mathbb{Z}$$

allows one to label such a facet by a residue  $j \bmod n$ ; call this facet  $F_{j \bmod n}$ . Then for any subset  $A \subseteq \{0, 1, \dots, \phi(n)\}$ , let  $K_A$  denote the subcomplex of  $K_{p_1, \dots, p_d}$  which is generated by the facets  $\{F_{j \bmod n}\}$  as  $j$  runs through the following set of residues:

$$A \cup \{\phi(n) + 1, \phi(n) + 2, \dots, n - 2, n - 1\}.$$

Our first main result interprets the magnitudes of the coefficients of  $\Phi_n(x)$ . Let  $\tilde{H}_i(-; \mathbb{Z})$  denote reduced simplicial homology with coefficients in  $\mathbb{Z}$ .

**Theorem 1.** *For a squarefree positive integer  $n = p_1 \cdots p_d$ , with cyclotomic polynomial  $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$ , one has*

$$\tilde{H}_i(K_{\{j\}}; \mathbb{Z}) = \begin{cases} \mathbb{Z}/c_j\mathbb{Z} & \text{if } i = d - 2, \\ \mathbb{Z} & \text{if both } i = d - 1 \text{ and } c_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It should be noted that, starting with  $d = 3$  and for large enough primes  $p_i$ , one can exhibit  $(d-1)$ -dimensional subcomplexes of  $K_{p_1, \dots, p_d}$  with the following properties: like  $K_{\{j\}}$ , their only nonvanishing homology group lies in dimension  $d-2$  and consists entirely of torsion, but unlike  $K_{\{j\}}$ , this torsion group need not be cyclic and can require arbitrarily many generators.

We furthermore interpret topologically the signs of the coefficients in  $\Phi_n(x)$ . For this, we use oriented simplicial homology, and orient the facet  $F_{j \bmod n}$  having  $j \equiv j_i \bmod p_i$  for  $i = 1, 2, \dots, d$  as

$$(3) \quad [F_j] = [F_{j \bmod n}] = [j_1 \bmod p_1, \dots, j_d \bmod p_d].$$

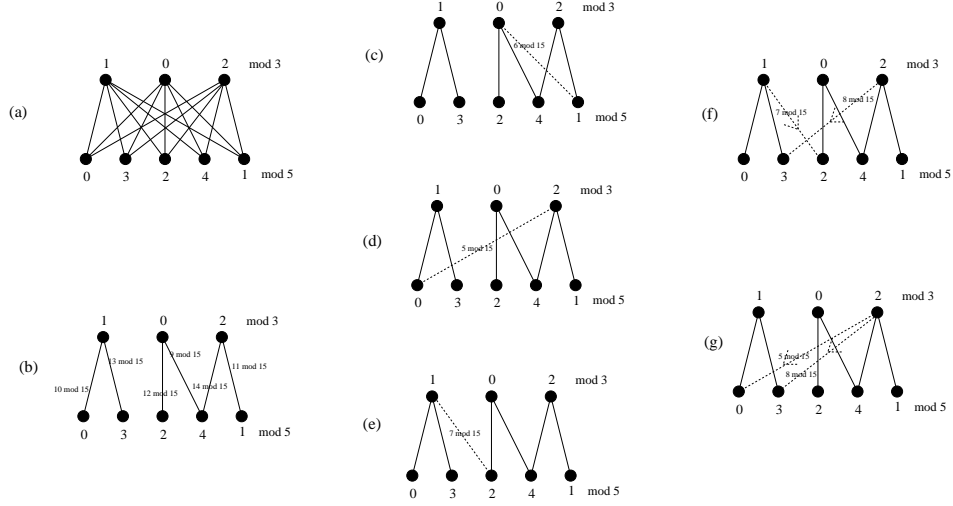
**Theorem 2.** *Fix a squarefree positive integer  $n = p_1 \cdots p_d$  with cyclotomic polynomial  $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$ . Then for any  $j \neq j'$  such that  $c_j, c_{j'} \neq 0$ , one has  $\tilde{H}_{d-1}(K_{\{j, j'\}}; \mathbb{Z}) \cong \mathbb{Z}$ , and any nonzero  $(d-1)$ -cycle  $z = \sum_{\ell} b_{\ell} [F_{\ell}]$  in this homology group will have  $b_j, b_{j'} \neq 0$ , with*

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

*In particular,  $c_j, c_{j'}$  have the same sign if and only if  $b_j, b_{j'}$  have opposite signs.*

**Example 3.** We illustrate Theorems 1 and 2 for  $n = 15$ . Here  $d = 2, p_1 = 3, p_2 = 5$ , and  $\phi(n) = 2 \cdot 4 = 8$ . The cyclotomic polynomial is

$$\begin{aligned} \Phi_{15}(x) &= 1 - x + x^3 - x^4 + x^5 - x^7 + x^8 \\ &= (+1) \cdot (x^0 + x^3 + x^5 + x^8) + (-1) \cdot (x^1 + x^4 + x^7) + 0 \cdot (x^2 + x^6). \end{aligned}$$

FIGURE 1. The case of  $\Phi_{15}(x)$ 

The complex  $K_{p_1, p_2} = K_{3,5}$  is a complete bipartite graph with vertex sets labelled as in Figure 1(a). The subcomplex  $K_{\emptyset}$  generated by the edges  $F_j \bmod 15$  with  $j \in \{\phi(n) + 1, \phi(n) + 2, \dots, n - 1\} = \{9, 10, 11, 12, 13, 14\}$  is the subgraph shown in Figure 1(b).

To see why the coefficient  $c_6 = 0$  in  $\Phi_{15}(x)$ , one adds the edge  $F_6 \bmod 15$  to the graph  $K_{\emptyset}$ , obtaining the graph  $K_{\{6\}}$ , shown in Figure 1(c), which has

$$\tilde{H}_0(K_{\{6\}}; \mathbb{Z}) = \mathbb{Z} = \mathbb{Z}/0\mathbb{Z}$$

$$\tilde{H}_1(K_{\{6\}}; \mathbb{Z}) = \mathbb{Z}.$$

To see why the coefficients  $c_5 = +1$  or  $c_7 = -1$  have magnitude 1, one adds the edge  $F_5 \bmod 15$  or  $F_7 \bmod 15$  to the graph  $K_{\emptyset}$ , obtaining the graphs  $K_{\{5\}}$  or  $K_{\{7\}}$  shown in Figures 1(d) and 1(e), which have

$$\tilde{H}_0(K_{\{5\}}; \mathbb{Z}) = 0 = \mathbb{Z}/(+1)\mathbb{Z}$$

$$\tilde{H}_0(K_{\{7\}}; \mathbb{Z}) = 0 = \mathbb{Z}/(-1)\mathbb{Z}.$$

To understand the signs of the coefficients, note first that, by convention,  $\Phi_{15}(x)$  is monic, so the coefficient  $c_8 = c_{\phi(n)} = +1$ . Therefore any other coefficient  $c_j$  should have sign

$$\text{sgn}(c_j) = \frac{\text{sgn}(c_j)}{\text{sgn}(c_8)} = -\frac{\text{sgn}(b_8)}{\text{sgn}(b_j)}$$

where  $z = \sum_i b_i [F_i]$  is a nontrivial cycle in  $K_{\{j,8\}}$ , in which the edge  $[F_j]$  is directed from the vertex  $(j_1 \bmod 3)$  toward the vertex  $(j_2 \bmod 5)$ . As shown in Figures 1(f) and 1(g), the nontrivial cycle in  $K_{\{7,8\}}$  has  $[F_7], [F_8]$  oriented in the *same* direction, explaining why  $c_7 = -1$ , while the nontrivial cycle in  $K_{\{5,8\}}$  has  $[F_5], [F_8]$  oriented in the *opposite* direction, explaining why  $c_5 = +1$ .

The remainder of the paper is structured as follows. Section 2 describes an initial interpretation for the cyclotomic polynomial, which applies much more generally to any monic polynomial in  $\mathbb{Z}[x]$ . Section 3 reviews some facts, underlying the main results, about duality of matroids, Plücker coordinates, and oriented matroids.

Section 4 recalls results and establishes terminology on Kalai's higher dimensional spanning trees in a simplicial complex. Section 5 discusses further properties of the simplicial complex  $K_{p_1, \dots, p_d}$  whose subcomplexes appear in Theorem 1 and 2. Section 6 prove these theorems. Section 7 gives some reformulations of Theorem 1 and 2 suggested to the authors by D. Fuchs. Section 8 explains how well-known properties of the cyclotomic polynomial manifest themselves topologically.

We remark that, since the submission of this paper for publication, an alternate proof of Theorem 1 has appeared in the work of R. Meshulam [21], using the *Fourier transform* on the group  $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_d}$ .

## 2. COEFFICIENTS OF MONIC POLYNOMIALS IN $\mathbb{Z}[x]$

Our goal here is an initial interpretation for the coefficients of  $\Phi_n(x)$ , which applies more generally to the coefficients of *any* monic polynomial  $f(x)$  in  $\mathbb{Z}[x]$ . Recall that when  $f(x)$  is of degree  $r$ , one has an isomorphism of  $\mathbb{Z}$ -modules

$$\begin{aligned} \mathbb{Z}^r &\longrightarrow \mathbb{Z}[x]/(f(x)) \\ (a_0, a_1, \dots, a_{r-1}) &\longmapsto \sum_{j=0}^{r-1} a_j \bar{x}^j. \end{aligned}$$

As notation, let  $R := \mathbb{Z}[x]/(f(x))$ , and for a subset  $A$  of some abelian group, let  $\mathbb{Z}A$  denote the collection of all  $\mathbb{Z}$ -linear combinations of elements of  $A$ .

**Proposition 4.** *For a monic polynomial  $f(x) = \sum_{j=0}^r c_j x^j$  of degree  $r$  in  $\mathbb{Z}[x]$ , one has an isomorphism of abelian groups*

$$R/\mathbb{Z}A \cong \mathbb{Z}/c_j \mathbb{Z}$$

where  $A$  is the subset of size  $r$  given as  $\{\bar{1}, \bar{x}, \bar{x}^2, \dots, \bar{x}^r\} \setminus \{\bar{x}^j\}$ .

*Proof.* Consider the matrix in  $\mathbb{Z}^{r \times (r+1)}$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -c_0 \\ 0 & 1 & 0 & \dots & 0 & -c_1 \\ 0 & 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -c_{r-1} \end{bmatrix}$$

whose columns express the elements of  $\{\bar{1}, \bar{x}, \bar{x}^2, \dots, \bar{x}^r\}$  uniquely in the  $\mathbb{Z}$ -basis  $\{\bar{1}, \bar{x}, \bar{x}^2, \dots, \bar{x}^{r-1}\}$  for  $R = \mathbb{Z}[x]/(f)$ . The  $r \times r$  submatrix obtained by restricting this matrix to the columns indexed by  $A$  is equivalent by row and column operations (invertible over  $\mathbb{Z}$ ) to a diagonal matrix with diagonal entries  $(1, 1, \dots, 1, -c_j)$ . Hence  $R/\mathbb{Z}A \cong \mathbb{Z}/c_j \mathbb{Z}$ .  $\square$

The special case where  $f(x)$  is the cyclotomic polynomial  $\Phi_n(x)$  leads to the following considerations. Fix once and for all a primitive  $n^{\text{th}}$  root of unity  $\zeta$ .

**Corollary 5.** *The cyclotomic polynomial  $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$  has*

$$\mathbb{Z}[\zeta]/\mathbb{Z}A \cong \mathbb{Z}/c_j \mathbb{Z}$$

where  $A = \{1, \zeta, \zeta^2, \dots, \zeta^{\phi(n)}\} \setminus \{\zeta^j\}$ .

*Proof.* Apply the previous proposition with  $f(x) = \Phi_n(x)$  and  $r = \phi(n)$ , noting that the ring map  $\mathbb{Z}[x] \rightarrow \mathbb{Z}[\zeta]$  sending  $x$  to  $\zeta$  will also send  $x^j$  to  $\zeta^j$ , and induce an isomorphism  $\mathbb{Z}[x]/(\Phi_n(x)) \rightarrow \mathbb{Z}[\zeta]$ .  $\square$

For later use (see the proof of Theorem 21), we prove here that the set

$$P_n := \{\zeta^m\}_{m \in (\mathbb{Z}/n\mathbb{Z})^\times}$$

of all *primitive*  $n^{\text{th}}$  roots of unity within  $\mathbb{Z}[\zeta]$  forms a  $\mathbb{Z}$ -basis whenever  $n$  is square-free. This is a sharpening of an observation of Johnsen [13], who noted that  $P_n$  forms a  $\mathbb{Q}$ -basis of  $\mathbb{Q}[\zeta]$  in the same situation.

**Proposition 6.** *When  $n$  is squarefree, the collection  $P_n$  of all primitive  $n^{\text{th}}$  roots of unity forms a  $\mathbb{Z}$ -basis for  $\mathbb{Z}[\zeta]$ .*

*Proof.* Note that  $|P_n| = \phi(n)$  and that  $\mathbb{Z}[\zeta]$  is a free abelian group of rank  $\phi(n)$ . Therefore it suffices to show that  $P_n$  is a  $\mathbb{Z}$ -spanning set for  $\mathbb{Z}[\zeta]$ .

This spanning is clear when  $n = p$  is prime since the only non-primitive root of unity is  $1 = -(\zeta + \zeta^2 + \cdots + \zeta^{p-1})$ .

When  $n = p_1 \cdots p_d$  for  $d > 1$ , temporarily use the notation  $\zeta_n$  for a fixed primitive  $n^{\text{th}}$  root of unity, and similarly for  $\zeta_{p_i}$ . The ring inclusions defined by

$$\begin{aligned} \mathbb{Z}[\zeta_{p_i}] &\xrightarrow{f_i} \mathbb{Z}[\zeta_n] \\ \zeta_{p_i} &\longmapsto (\zeta_n)^{\frac{n}{p_i}} \end{aligned}$$

assemble to give a  $\mathbb{Z}$ -module map  $f := f_1 \otimes \cdots \otimes f_d$

$$(4) \quad \begin{aligned} \mathbb{Z}[\zeta_{p_1}] \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_{p_d}] &\longrightarrow \mathbb{Z}[\zeta_n] \\ \zeta_{p_1}^{a_1} \otimes \cdots \otimes \zeta_{p_d}^{a_d} &\longmapsto \zeta_n^j \end{aligned}$$

where  $j \equiv \sum_{i=1}^d a_i \frac{n}{p_i} \pmod{n}$ . Note that

$$\begin{aligned} \frac{n}{p_i} &\equiv 0 \pmod{p_{i_0}} \text{ for } i \neq i_0, \\ \frac{n}{p_{i_0}} &\not\equiv 0 \pmod{p_{i_0}}. \end{aligned}$$

Consequently, for each  $i_0 = 1, 2, \dots, d$ , one has  $j \equiv a_{i_0} \frac{n}{p_{i_0}} \pmod{p_{i_0}}$ , and as  $a_{i_0}$  runs over  $\mathbb{Z}/p_{i_0}\mathbb{Z}$ , the product  $a_{i_0} \frac{n}{p_{i_0}}$  does the same. The map  $f$  is then surjective by the Chinese Remainder Theorem isomorphism (2).

Lastly, note that the source of  $f$  has a  $\mathbb{Z}$ -basis consisting of those  $\zeta_{p_1}^{a_1} \otimes \cdots \otimes \zeta_{p_d}^{a_d}$  in which each  $a_i \not\equiv 0 \pmod{p_i}$ . Since this means that each  $a_i \frac{n}{p_i} \not\equiv 0 \pmod{p_i}$ , this basis maps under  $f$  to the set  $P_n$ . Hence  $P_n$  must  $\mathbb{Z}$ -linearly span the target  $\mathbb{Z}[\zeta]$ , and furthermore form a  $\mathbb{Z}$ -basis for the target.  $\square$

### 3. DUALITY OF MATROIDS OR PLÜCKER COORDINATES

We will need a version of the linear algebraic duality between Plücker coordinates for complementary Grassmannians  $G(r, \mathbb{F}^n)$ ,  $G(n-r, \mathbb{F}^n)$ , or equivalently, the duality between bases and cobases in coordinatized matroids.

**Proposition 7.** *Let  $0 \leq r \leq n$ . Let  $M$  and  $M^\perp$  be matrices in  $\mathbb{F}^{r \times n}$  and  $\mathbb{F}^{(n-r) \times n}$ , respectively, both of maximal rank, with the following property:  $\ker M$  is equal to the row space of  $M^\perp$ , or equivalently,  $\ker M^\perp$  is the row space of  $M$ . Then*

- (i) *there exists a scalar  $\alpha$  in  $\mathbb{F}^\times$  having the following property: for every  $(n-r)$ -subset  $T$  of  $[n]$ , with complementary set  $T^c$ ,*

$$\det(M|_{T^c}) = \pm\alpha \cdot \det(M^\perp|_T)$$

*where  $A|_J$  denotes the restriction of a matrix  $A$  to the subset of columns indexed by  $J$ , and where the  $\pm$  sign depends upon the set  $T$ .*

- (ii) *if one furthermore assumes that  $\mathbb{F} = \mathbb{Q}$ , that  $M$  and  $M^\perp$  have entries in  $\mathbb{Z}$ , and that there exists at least one  $(n-r)$ -subset  $T_0$  for which  $M|_{T_0^c}, M^\perp|_{T_0}$  are both invertible over  $\mathbb{Z}$ , then the scalar  $\alpha$  above equals  $\pm 1$ , and one has for every  $(n-r)$ -subset  $T$ ,*

$$\text{coker}(M|_{T^c}) \cong \text{coker}(M^\perp|_T).$$

*Here we are regarding  $M|_{T^c}$  and  $M^\perp|_T$  as operators on  $\mathbb{Z}^r$  and  $\mathbb{Z}^{n-r}$ , respectively.*

*Proof.* For assertion (i), note that the hypotheses and conclusions are unchanged if one performs row operations over  $\mathbb{F}$  separately on  $M, M^\perp$ , and if one permutes simultaneously the columns of  $M, M^\perp$  by the same permutation of  $[n]$ .

Thus one can assume without loss of generality that the full rank matrix  $M$  has the form  $M = [I_r \mid A]$  for some matrix  $A$  in  $\mathbb{F}^{r \times (n-r)}$ . In this case, the kernel of  $M$  is spanned by the rows of  $[-A^t \mid I_{n-r}]$ , and hence by performing row operations on  $M^\perp$ , one can assume without loss of generality that  $M^\perp = [-A^t \mid I_{n-r}]$ .

When  $M, M^\perp$  have these special forms, one has that

$$\begin{aligned} M|_{T^c} &= \left[ \begin{array}{c|c} I_r|_{[r] \setminus T} & A|_{[n] \setminus ([r] \cup T)} \end{array} \right] \\ M^\perp|_T &= \left[ \begin{array}{c|c} -A^t|_{[r] \cap T} & I_{n-r}|_T \end{array} \right]. \end{aligned}$$

Hence  $\det(M|_{T^c})$  will be, up to sign, the determinant of  $A$  restricted to its columns in  $[n] \setminus ([r] \cup T)$ , and to its rows in  $[r] \cap T$ , while  $\det(M^\perp|_T)$  will be, up to sign, the determinant of  $-A^t$  restricted to its columns in  $[r] \cap T$ , and to its rows in  $[n] \setminus ([r] \cup T)$ . These determinants are the same up to sign.

For assertion (ii), again the hypotheses and conclusion are unchanged if one performs row operations *invertible over  $\mathbb{Z}$*  to  $M, M^\perp$ , and if one permutes columns simultaneously in  $M, M^\perp$ . Thus without loss of generality, one can assume that

$$\begin{aligned} T^c &= [r] \\ T &= [n] \setminus [r] \\ M &= [I_r \mid A] \\ M^\perp &= [-A^t \mid I_{n-r}] \end{aligned}$$

as above. But in this case, one can check that  $\text{coker}(M|_{T^c}) \cong \text{coker}(M^\perp|_T)$  for the same reason that their determinants agree up to sign: their cokernels are isomorphic to the cokernels of matrices which are, up to sign, the transposes of each other.  $\square$

The proof of Theorem 2 will ultimately rely on the following statement about duality of *oriented matroids* for vectors over an ordered field  $\mathbb{F}$ , such as  $\mathbb{F} = \mathbb{Q}$ .

**Proposition 8.** *Let  $\mathbb{F}$  be an ordered field,  $M$  and let  $M^\perp$  be matrices in  $\mathbb{F}^{r \times n}$  and  $\mathbb{F}^{(n-r) \times n}$  as in Proposition 7, that is, both of maximal rank, with  $\ker M$  equal to*

the row space of  $M^\perp$ . Let the vectors  $v_\ell$  in  $\mathbb{F}^r$  and  $v_\ell^\perp$  in  $\mathbb{F}^{n-r}$  be the  $\ell^{\text{th}}$  columns of  $M$  and  $M^\perp$ .

Let  $A$  be an  $(r+1)$ -subset of  $\{1, 2, \dots, n\}$  such that the matrix  $M|_A$  in  $\mathbb{F}^{r \times (r+1)}$  has full rank  $r$ , with

$$(5) \quad \sum_{\ell \in A} c_\ell v_\ell = 0$$

the unique dependence among its columns, up to scaling.

Then for any pair of nonzero coefficients  $c_j, c_{j'} \neq 0$ , the matrix  $M^\perp|_{A^c \cup \{j, j'\}}$  in  $\mathbb{F}^{(n-r) \times (n-r+1)}$  has full rank  $n-r$ , and the unique dependence among its columns, up to scaling,

$$(6) \quad \sum_{\ell \in A^c \cup \{j, j'\}} b_\ell v_\ell^\perp = 0,$$

will have both  $b_j, b_{j'} \neq 0$ , with

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

In particular,  $c_j, c_{j'}$  have the same sign if and only if  $b_j, b_{j'}$  have opposite signs.

*Proof.* First let us show the assertion about ranks. Since  $M|_A$  has rank  $r$ , the fact that  $c_{j'} \neq 0$  implies that  $v_{j'}$  lies in the span of the columns of  $M|_{A \setminus \{j'\}}$ . Hence the matrix  $M|_{A \setminus \{j'\}}$  in  $\mathbb{F}^{r \times r}$  has rank  $r$ , and its columns give a basis for  $\mathbb{F}^r$ . Then Proposition 7(i) implies that the columns of  $M^\perp|_{A^c \cup \{j'\}}$  form a basis of  $\mathbb{F}^{n-r}$ , and thus  $M^\perp|_{A^c \cup \{j, j'\}}$  has rank  $n-r$ . This also shows that  $v_{j'}^\perp$  is dependent on the remaining columns of  $M^\perp|_{A^c \cup \{j, j'\}}$ , so that (6) will at least have  $b_{j'} \neq 0$ .

Now a dependence (5) extends by zeroes to a vector  $(c_1, \dots, c_{r+1}, 0, \dots, 0)$  in  $\mathbb{F}^n$  that lies in  $\ker(M)$ , and hence also lies in the row space of  $M^\perp$ . However, vectors in the row space of  $M^\perp$  are *covectors* for  $\{v_\ell^\perp\}$  in the sense that they give the values of linear functionals  $f$  in  $(\mathbb{F}^{n-r})^*$  when applied to the list of vectors  $(v_1^\perp, \dots, v_n^\perp)$ . Thus there is a functional  $f$  having  $f(v_\ell^\perp) = c_\ell$  for  $\ell \in A$  and  $f(v_\ell^\perp) = 0$  for  $\ell \in A^c$ . Applying this  $f$  to (6) gives  $c_j b_j + c_{j'} b_{j'} = 0$  which is equivalent to the remaining assertion of the proposition.  $\square$

#### 4. SIMPLICIAL SPANNING TREES

For a collection of subsets  $S$  of some vertex set  $V$ , let  $\langle S \rangle$  denote the (abstract) simplicial complex  $S$  on  $V$  generated by  $S$ , that is,  $\langle S \rangle \subset 2^V$  consists of all subsets of  $V$  contained in at least one subset from  $S$ . We recall the notion of a simplicial spanning tree in  $S$ , following Adin [1], Duval, Klivans and Martin [8], Kalai [14], and Maxwell [20].

**Definition 9.** Let  $S$  be the collection of facets of a pure  $k$ -dimensional (abstract) simplicial complex. Say that  $R \subset S$  is an  $S$ -spanning tree if

- (i)  $\langle R \rangle$  contains the entire  $(k-1)$ -skeleton of  $\langle S \rangle$ ,
- (ii)  $\tilde{H}_k(\langle R \rangle; \mathbb{Z}) = 0$ , and
- (iii)  $\tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z})$  is finite.

We point out here three well-known features of this definition.

**Proposition 10.** *Fix the collection of facets  $S$  of a pure  $k$ -dimensional simplicial complex.*

- (i) *Condition (i) in Definition 9 is equivalent to  $\tilde{H}_k(\langle S \rangle, \langle R \rangle; \mathbb{Z}) = \mathbb{Z}^{|S \setminus R|}$ .*
- (ii) *Condition (ii) in Definition 9 is equivalent to  $\tilde{H}_k(\langle R \rangle; \mathbb{Q}) = 0$ .*
- (iii) *All  $S$ -spanning trees  $R$  have the same cardinality, namely*

$$(7) \quad |R| = |S| - \text{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z}).$$

*Proof.* **Proof of (i).** Note that  $\tilde{H}_k(\langle S \rangle, \langle R \rangle; \mathbb{Z}) = \tilde{Z}_k(\langle S \rangle, \langle R \rangle; \mathbb{Z})$ , since  $\langle S \rangle$  is  $k$ -dimensional. By definition, the relative cycle group  $\tilde{Z}_k(\langle S \rangle, \langle R \rangle; \mathbb{Z})$  equals the kernel of the map

$$\partial_k : \tilde{C}_k(\langle S \rangle, \langle R \rangle; \mathbb{Z}) \rightarrow \tilde{C}_{k-1}(\langle S \rangle, \langle R \rangle; \mathbb{Z}).$$

Because both the source and target of  $\partial_k$  are free abelian, with the source of rank  $|S \setminus R|$ , its kernel is free abelian of the same rank if and only if  $\partial_k$  is the zero map. This last condition is equivalent to Condition (i) in Definition 9.

**Proof of (ii).** This follows from the fact that  $\langle R \rangle$  is  $k$ -dimensional, so that

$$\begin{aligned} \tilde{H}_k(\langle R \rangle; \mathbb{Z}) &= \tilde{Z}_k(\langle R \rangle; \mathbb{Z}), \text{ and} \\ \tilde{H}_k(\langle R \rangle; \mathbb{Q}) &= \tilde{Z}_k(\langle R \rangle; \mathbb{Q}). \end{aligned}$$

The cycle group  $\tilde{Z}_k(\langle R \rangle; \mathbb{Z})$  (respectively,  $\tilde{Z}_k(\langle R \rangle; \mathbb{Q})$ ) vanishes if and only if the collection of boundaries of simplices in  $R$  are linearly independent over  $\mathbb{Z}$  (respectively, over  $\mathbb{Q}$ ). However, these two notions of linear independence are equivalent.

**Proof of (iii).** Consider this portion of the long exact sequence of the pair  $(\langle S \rangle, \langle R \rangle)$

$$\begin{array}{ccccccc} \tilde{H}_k(\langle R \rangle; \mathbb{Z}) & \rightarrow & \tilde{H}_k(\langle S \rangle; \mathbb{Z}) & \rightarrow & \tilde{H}_k(\langle S \rangle, \langle R \rangle; \mathbb{Z}) & \rightarrow & \tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z}) \\ \parallel & & & & \parallel & & \\ 0 & & & & \mathbb{Z}^{|S \setminus R|} & & \end{array}.$$

Here the two vertical equalities come from Condition (ii) in Definition 9 and from assertion (i) above, respectively. Since the last term  $\tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z})$  in this sequence is a finite abelian group by Condition (iii) in Definition 9, the sequence shows that  $\tilde{H}_k(\langle S \rangle; \mathbb{Z})$  is a subgroup of  $\mathbb{Z}^{|S \setminus R|}$  of finite index. Consequently, it must be free abelian of the same rank. Hence  $|S \setminus R| = \text{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z})$  and equation (7) follows.  $\square$

The following observation essentially goes back to work of Kalai [14, Lemma 2].

**Proposition 11.** *Fix a vertex set  $V$  and a collection of  $k$ -dimensional simplices  $S$ . Consider a collection of  $(k+1)$ -dimensional faces  $T$  of cardinality*

$$|T| := \text{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z})$$

*for which  $T \cup \langle S \rangle$  forms a simplicial complex  $K$ , that is, all boundaries of faces in  $T$  lie in  $\langle S \rangle$ .*

*Then the following two assertions hold for any choice of an  $S$ -spanning tree  $R$ .*

- (i) *The  $|T| \times |T|$  matrix  $\partial$  that represents the relative simplicial boundary map*

$$\begin{array}{ccc} C_{k+1}(K, \langle R \rangle; \mathbb{Z}) & \rightarrow & C_k(K, \langle R \rangle; \mathbb{Z}) \\ \parallel & & \parallel \\ \mathbb{Z}^{|T|} & & \mathbb{Z}^{|S \setminus R|} \end{array}$$

*is nonsingular if and only if  $\tilde{H}_{k+1}(K; \mathbb{Q}) = 0$ .*



(ii) When the matrix  $\partial$  is nonsingular, then  $\text{coker}(\partial) = \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$ .

*Proof.* **Proof of (i).** Start by noting that  $\tilde{H}_k(\langle R \rangle; \mathbb{Q}) = 0$  due to part (ii) of Proposition 10. Since  $\langle R \rangle$  is only  $k$ -dimensional, it also has  $\tilde{H}_{k+1}(\langle R \rangle; \mathbb{Q}) = 0$ . Consequently,

$$\tilde{H}_{k+1}(K; \mathbb{Q}) \cong \tilde{H}_{k+1}(K, \langle R \rangle; \mathbb{Q}) = Z_{k+1}(K, \langle R \rangle; \mathbb{Q}),$$

where the isomorphism comes from the long exact sequence in homology for the pair  $(K, \langle R \rangle)$ , and the equality follows since  $K$  is  $(k+1)$ -dimensional. Therefore  $\tilde{H}_{k+1}(K; \mathbb{Q})$  vanishes if and only if  $Z_{k+1}(K, \langle R \rangle; \mathbb{Q})$  vanishes, which occurs if and only the square matrix  $\partial$  has vanishing kernel, i.e. it is nonsingular.

**Proof of (ii).** We first note that  $C_k(K, \langle R \rangle; \mathbb{Z}) = Z_k(K, \langle R \rangle; \mathbb{Z})$  due to these facts:

- (a) every  $k$ -simplex in  $K$  actually lies in  $S$  by our assumption on  $T$ , and
- (b) every boundary of a  $k$ -simplex in  $S$  lies in  $R$  by Condition (i) in Definition 9.

Thus when  $\partial$  is nonsingular, one has

$$\begin{aligned} \text{coker}(\partial) &= C_k(K, \langle R \rangle; \mathbb{Z}) / B_k(K, \langle R \rangle; \mathbb{Z}) \\ &= Z_k(K, \langle R \rangle; \mathbb{Z}) / B_k(K, \langle R \rangle; \mathbb{Z}) = \tilde{H}_k(K, \langle R \rangle; \mathbb{Z}). \end{aligned}$$

□

**Definition 12.** Given a collection of  $k$ -simplices  $S$ , and an  $S$ -spanning tree  $R$ , say<sup>2</sup> that  $R$  is *torsion-free* if Condition (iii) in Definition 9 is strengthened to the vanishing condition

$$(iv) \quad \tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z}) = 0.$$

**Example 13.** For example, when  $\langle R \rangle$  is a contractible subcomplex of  $\langle S \rangle$  then it satisfies Condition (ii) of Definition 9 as well as the vanishing condition (iv). If it furthermore satisfies Condition (i) of Definition 9, then  $R$  becomes a torsion-free  $S$ -spanning tree.

A frequent combinatorial setting where this occurs (such as in Proposition 15 below) is when  $S$  is the set of facets of a (pure) *shellable* [3] simplicial complex, and  $R$  is the subset of facets which are not fully attached along their entire boundaries during the shelling process.

**Proposition 14.** Using the hypotheses and notation of Proposition 11, if one assumes in addition that  $R$  is torsion-free, assertion (ii) of Proposition 11 becomes the following assertion about (non-relative) homology:

(ii) When the matrix  $\partial$  is nonsingular, then  $\text{coker}(\partial) = \tilde{H}_k(K; \mathbb{Z})$

*Proof.* When  $R$  is torsion-free, the long exact sequence for the pair  $(K, \langle R \rangle)$  shows that  $\tilde{H}_k(K; \mathbb{Z}) \cong \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$ . □

## 5. MORE ON THE COMPLETE $d$ -PARTITE COMPLEX

It is well-known and easy to see that for a positive integer  $n$  having prime factorization  $n = p_1^{e_1} \cdots p_d^{e_d}$  with  $e_i \geq 1$ , one has  $\Phi_n(x) = \Phi_{p_1 \cdots p_d}(x^{n/p_1 \cdots p_d})$ . Thus it suffices to interpret the coefficients of cyclotomic polynomials for squarefree  $n$ .

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<sup>2</sup>This condition on an  $S$ -spanning tree also plays an important role in forthcoming work by Duval, Klivans and Martin [9].

In this section, we fix such a squarefree  $n = p_1 \cdots p_d$ , and discuss further properties of the simplicial complexes  $K_{p_1, \dots, p_d}$ , defined in Section 1, appearing in Theorems 1 and 2.

**Proposition 15.** *The  $(d-2)$ -dimensional skeleton of  $K_{p_1, \dots, p_d}$  is shellable, with*

$$\tilde{H}_{d-2}(K_{p_1, \dots, p_d}; \mathbb{Z}) = \mathbb{Z}^{n-\phi(n)}.$$

*Proof.* To show that the  $(d-2)$ -skeleton is shellable, we note the following three facts: (i) zero-dimensional complexes are all trivially shellable, (ii) joins of shellable complexes are shellable [25, Sec. 2], and (iii) skeleta of (pure) shellable simplicial complexes are shellable [5, Corollary 10.12]. Having shown that this skeleton is shellable, it therefore has only top homology; see, for example [3, Appendix]. This homology is free abelian, of rank equal to the absolute value of its reduced Euler characteristic, namely

$$\begin{aligned} \left| \sum_{i \geq -1} (-1)^i \text{rank}_{\mathbb{Z}}(C_i) \right| &= \left| \sum_{i \geq -1} (-1)^i \sum_{\substack{I \subseteq \{1, 2, \dots, d\} \\ |I|=i+1}} \prod_{i \in I} p_i \right| \\ &= \left| \sum_{I \subsetneq \{1, 2, \dots, d\}} (-1)^{|I|-1} \prod_{i \in I} p_i \right| \\ &= |(p_1 - 1) \cdots (p_d - 1) - p_1 \cdots p_d| \\ &= |\phi(n) - n|. \end{aligned}$$

□

As noted in the introduction, the Chinese Remainder Theorem isomorphism (2) identifies elements of  $\mathbb{Z}/n\mathbb{Z}$  with the  $(d-1)$ -dimensional simplices of  $K_{p_1, \dots, p_d}$ . Lower dimensional faces of  $K_{p_1, \dots, p_d}$  can also be identified as cosets of subgroups within  $\mathbb{Z}/n\mathbb{Z}$ , but we will use this identification sparingly in this paper. For the sake of writing down oriented simplicial boundary maps, choose the following orientation on the simplices of  $K_{p_1, \dots, p_d}$ , consistent with the orientation of facets preceding Theorem 2: choose the oriented  $(\ell-1)$ -simplex  $[j_{i_1} \bmod p_{i_1}, \dots, j_{i_\ell} \bmod p_{i_\ell}]$  with  $i_1 < \dots < i_\ell$  as a basis element of  $C_{\ell-1}(K_{p_1, \dots, p_d}; \mathbb{Z})$ . The following simple observation was the crux of the results in [19].

**Proposition 16.** *If one identifies the indexing set  $\mathbb{Z}/n\mathbb{Z}$  for the columns of the boundary map*

$$(8) \quad C_{d-1}(K_{p_1, \dots, p_d}; \mathbb{Z}) \rightarrow C_{d-2}(K_{p_1, \dots, p_d}; \mathbb{Z})$$

*with the set  $\mu_n := \{\zeta^j\}_{j \in \mathbb{Z}/n\mathbb{Z}}$  of all  $n^{\text{th}}$  roots of unity, then every row of this boundary map represents a  $\mathbb{Q}$ -linear dependence on  $\mu_n$ .*

*Proof.* A row in this boundary map is indexed by an oriented  $(d-2)$ -face, which has the form  $[j_1 \bmod p_1, \dots, j_k \bmod p_k, \dots, j_d \bmod p_d]$  for some  $j_k \in \{0, 1, \dots, p_k - 1\}$  and  $1 \leq k \leq d$ . This row will then contain all zeroes except for entries equal to

$(-1)^{k-1}$  in the columns indexed by  $\zeta^j$  where

$$\begin{aligned} j &\equiv j_1 \pmod{p_1}, \\ &\vdots \\ j &\equiv j_d \pmod{p_d} \end{aligned}$$

except that  $j \pmod{p_k}$  is allowed to be arbitrary. These exponents  $j$  are exactly those lying in one coset of the subgroup  $p_1 \cdots \hat{p}_k \cdots p_d \mathbb{Z} / n\mathbb{Z}$  within  $\mathbb{Z} / n\mathbb{Z}$ . Summing  $\zeta^j$  over  $j$  in such a coset gives zero.  $\square$

**Example 17.** Let  $n = 15$  as in Example 3, and consider the matrix for the simplicial boundary map  $C_1(K_{3,5}; \mathbb{Z}) \rightarrow C_0(K_{3,5}; \mathbb{Z})$ . One of its rows is indexed by the 0-face  $[2 \pmod{5}]$  and this row has exactly three nonzero entries, all equal to  $(-1)^0 = +1$ . To see these signs, we rewrite  $[2 \pmod{5}]$  in three ways, all of which involve deleting the first entry out of two in an oriented 1-face:

$$[2 \pmod{5}] = [\widehat{0 \pmod{3}, 2 \pmod{5}}] = [\widehat{1 \pmod{3}, 2 \pmod{5}}] = [\widehat{2 \pmod{3}, 2 \pmod{5}}].$$

The columns corresponding to these three 1-faces are indexed by the roots of unity  $\zeta^{12}$ ,  $\zeta^7$ , and  $\zeta^2$ , respectively. Summing these up with coefficients of positive one, we get

$$1 \cdot \zeta^{12} + 1 \cdot \zeta^7 + 1 \cdot \zeta^2 = \zeta^2(\zeta^{10} + \zeta^5 + 1),$$

which is the sum of  $\zeta^j$  over  $j$  lying in a coset of  $5\mathbb{Z}/15\mathbb{Z}$ , and hence is zero.

**Definition 18.** Assume that  $n$  is squarefree and let  $T$  denote any set of  $n - \phi(n)$  columns of the boundary map (8). Identify the complementary set  $T^c$  of  $\phi(n)$  columns with a subset of the  $n^{\text{th}}$  roots-of-unity  $\mu_n$ . Create a subcomplex of  $K_{p_1, \dots, p_d}$  by including its entire  $(d-2)$ -skeleton and attaching the subset of  $(d-1)$ -faces indexed by  $T$ . We denote this subcomplex as  $K[T]$ .

Recall from the introduction that for any subset  $A \subseteq \{0, 1, \dots, \phi(n)\} \subset \mathbb{Z}/n\mathbb{Z}$ , we let  $K_A$  denote the subcomplex of  $K_{p_1, \dots, p_d}$  generated by the facets  $\{F_j \pmod{n}\}$  as  $j$  runs through the set of residues  $A \sqcup A_0$  where

$$A_0 := \{\phi(n) + 1, \phi(n) + 2, \dots, n - 2, n - 1\}.$$

**Proposition 19.** *The subcomplex  $K_\emptyset$ , and hence every subcomplex  $K_A$ , contains the full  $(d-2)$ -skeleton of  $K_{p_1, \dots, p_d}$ . Consequently,  $K_A = K[A \sqcup A_0]$ .*

*Proof.* Since a  $(d-2)$ -face of  $K_{p_1, \dots, p_d}$  corresponds to a coset  $j_0 + \frac{n}{p_i} (\mathbb{Z}/n\mathbb{Z})$  within  $\mathbb{Z}/n\mathbb{Z}$  for some  $i = 1, 2, \dots, d$  and  $j_0 \in \mathbb{Z}/n\mathbb{Z}$ , one must check that every such coset intersects  $A_0$ . As  $A_0$  is a consecutive sequence of  $n - \phi(n)$  residues, this amounts to checking that

$$n - \phi(n) \geq \frac{n}{p_i} \quad \text{for } i = 1, 2, \dots, d,$$

or equivalently, that

$$n \left(1 - \frac{1}{p_i}\right) \geq \phi(n) = n \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_d}\right).$$

This inequality holds since each factor in parenthesis on the right is less than 1.  $\square$

We next point out an interesting feature of the labelling of the boundary map (8) with regard to the set  $P_n$  of primitive  $n^{\text{th}}$  roots of unity, noted already in [19, Remark 5]. Let  $P_n^c$  denote the  $(n - \phi(n))$ -element subset of  $\mu_n$  indexed by the  $n^{\text{th}}$  roots of unity which are not primitive.

**Proposition 20.** *Let  $n$  be a squarefree integer and  $P_n^c$  be as above. Then the subcomplex  $K[P_n^c]$  of  $K_{p_1, \dots, p_d}$  is contractible.*

*Proof.* Observe that the primitive roots in  $\mathbb{Z}/n\mathbb{Z}$  are exactly those elements which do not vanish modulo  $p_i$  for  $i = 1, \dots, d$ . Tracing through the labelling of the  $(d - 1)$ -faces via  $\Xi$ , we obtain the description

$$K[P_n^c] = \bigcup_{i=1}^d \text{star}_{K_{p_1, \dots, p_d}}(0 \bmod p_i),$$

where  $\text{star}_{\Delta}(v)$  denotes the *simplicial star* of the vertex  $v$  inside a simplicial complex  $\Delta$ . Furthermore, each intersection of these stars is nonempty and contractible, because it is the star of another face: for  $I \subset [d]$ ,

$$\bigcap_{i \in I} \text{star}_{K_{p_1, \dots, p_d}}(0 \bmod p_i) = \text{star}_{K_{p_1, \dots, p_d}}(\{0 \bmod p_i\}_{i \in I}).$$

A standard nerve lemma [4, Theorem 10.6] then shows that  $K[P_n^c]$  itself is contractible. This also follows by induction on  $d$ , where the case  $d = 2$  appears as [12, Exercise 0.23, p. 20].  $\square$

**Theorem 21.** *Let  $n$  be a squarefree integer and  $T$  be a subset of  $\mu_n$  of size  $n - \phi(n)$ . Let  $K[T]$  be the subcomplex of  $K_{p_1, \dots, p_d}$  of Definition 18. Then*

$$\tilde{H}_i(K[T]; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[\zeta]/\mathbb{Z}T^c & \text{if } i = d - 2, \\ \mathbb{Z} & \text{if both } i = d - 1 \text{ and } \text{rank}_{\mathbb{Z}}(\mathbb{Z}T^c) < \phi(n), \\ 0 & \text{otherwise.} \end{cases}$$

where  $\mathbb{Z}T^c$  is the sublattice  $\mathbb{Z}$ -spanned by the roots-of-unity  $T^c \subset \mu_n$ .

*Proof.* Choose any  $\mathbb{Z}$ -basis for  $\mathbb{Z}[\zeta]$ . Let  $M$  in  $\mathbb{Z}^{\phi(n) \times n}$  be the matrix that expresses the  $n^{\text{th}}$  roots of unity  $\mu_n$  in this basis.

We construct a particular matrix  $M^\perp$  to accompany  $M$  as in Proposition 7 part (ii). Consider the collection  $S$  of all  $(d - 2)$ -faces in the complete  $d$ -partite complex  $K_{p_1, \dots, p_d}$ . The complex  $\langle S \rangle$  generated by  $S$  is therefore the  $(d - 2)$ -skeleton of  $K_{p_1, \dots, p_d}$ . Proposition 15 implies that  $\langle S \rangle$  is shellable, and that it has  $\text{rank}_{\mathbb{Z}} \tilde{H}_{d-2}(\langle S \rangle; \mathbb{Z}) = n - \phi(n)$ . Therefore, we are in the situation of Example 13, implying that there exists a torsion-free  $S$ -spanning tree  $R$ , and any such  $R$  will have  $|S \setminus R| = n - \phi(n)$ .

Our candidate for the matrix  $M^\perp$  in  $\mathbb{Z}^{(n - \phi(n)) \times n}$  is the restriction of the boundary map from (8) to its rows indexed by  $S \setminus R$ . Proposition 16 shows that the rows of  $M^\perp$  are all perpendicular to the rows of  $M$ .

Now choose  $T_0$  so that  $T_0^c$  indexes the set  $P_n$  of primitive  $n^{\text{th}}$  roots of unity. Proposition 6 implies that the maximal minor  $M|_{T_0^c}$  of  $M$  is invertible over  $\mathbb{Z}$ , while Proposition 20 implies that the maximal minor  $M^\perp|_{T_0}$  of  $M^\perp$  is invertible over  $\mathbb{Z}$ . Thus  $M, M^\perp$  satisfy the hypotheses of Proposition 7 part (ii), and combining this with Proposition 14 gives the assertion of the theorem.  $\square$

## 6. PROOF OF THEOREMS 1 AND 2

We are now in a position to prove Theorems 1 and 2.

*Proof of Theorem 1.* Let  $T^c = \{1, \zeta, \zeta^2, \dots, \zeta^{\phi(n)}\} \setminus \{\zeta^j\}$ , so that  $T = A_0 \cup \{j\}$ , and  $K[T] = K[A_0 \cup \{j\}] = K_{\{j\}}$  by Proposition 19. The theorem then follows from Theorem 21 and Corollary 5.  $\square$

*Proof of Theorem 2.* We prove Theorem 2 by applying Proposition 8 to the matrices  $M, M^\perp$  in the proof of Theorem 21, with  $A = \{1, \zeta, \zeta^2, \dots, \zeta^{\phi(n)}\}$ . Thus  $A^c = A_0$  and  $K_{\{j, j'\}} = K[A^c \cup \{j, j'\}]$  by Proposition 19.

The dependence (5) among the columns of  $M|_A$  has the same coefficients (up to scaling) as the cyclotomic polynomial, and the dependence (6) among the columns of  $M^\perp|_{A^c \cup \{j, j'\}}$  will have the same coefficients (up to scaling) as a nonzero cycle  $z = \sum_\ell b_\ell [F_\ell]$  in  $\tilde{H}_{d-1}(K_{\{j, j'\}}; \mathbb{Z})$ .  $\square$

## 7. ATTACHING MAP REFORMULATION

The authors are indebted to Dmitry Fuchs for suggesting reformulations of Theorems 1 and 2, explaining how the complete  $d$ -partite complex  $K_{p_1, \dots, p_d}$  is built from its subcomplex  $K_\emptyset$ , by attaching the facets  $F_{j \bmod n}$  along their boundaries. We briefly discuss this here, beginning with a homological version. Throughout this section, all homology groups are reduced, and taken with coefficients in  $\mathbb{Z}$ .

Recall from the Introduction that for  $j$  in  $\mathbb{Z}/n\mathbb{Z}$ , the facet  $F_{j \bmod n}$  was given a particular orientation  $[F_{j \bmod n}]$  as a basis element in the oriented reduced  $(d-1)$ -chains of  $K_{p_1, \dots, p_d}$ . Let  $[z_{j \bmod n}] := \partial[F_{j \bmod n}]$  denote the  $(d-2)$ -cycle which is its image under the simplicial boundary map  $\partial$ .

Letting  $\mathbb{S}^d, \mathbb{B}^d$  denote the  $d$ -dimensional sphere and ball respectively, denote by  $\mathbb{S}^{d-1} \cup_f \mathbb{B}^d$  the space obtained by attaching  $\mathbb{B}^d$  to  $\mathbb{S}^{d-1}$  along its boundary  $\text{Bd}(\mathbb{B}^d)$  via a map  $\text{Bd}(\mathbb{B}^d) \xrightarrow{f} \mathbb{S}^{d-1}$ . Recall (see, e.g. [12, pp. 12-13, §2.2, and Cor. 4.25]) that the homotopy type of  $\mathbb{S}^{d-1} \cup_f \mathbb{B}^d$  is determined by the absolute value of  $\deg(f)$ , the scalar defined by the map on the top homology groups

$$\tilde{H}_{d-1}(\text{Bd}(\mathbb{B}^d)) \cong \mathbb{Z} \xrightarrow{f_*} \mathbb{Z} \cong \tilde{H}_{d-1}(\mathbb{S}^d).$$

**Theorem 22.** *Let  $n = p_1 \cdots p_d$  be squarefree.*

(i) *One has a homology isomorphism*

$$\tilde{H}_*(K_\emptyset) \cong \tilde{H}_*(\mathbb{S}^{d-2}),$$

*with  $\tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$  generated by the cycle  $[z_{\phi(n) \bmod n}]$ .*

(ii) *If  $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$ , then for  $j = 0, 1, \dots, \phi(n)$ , one has*

$$[z_{j \bmod n}] = c_j [z_{\phi(n) \bmod n}] \quad \text{in} \quad \tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z},$$

*and a homology isomorphism*

$$\tilde{H}_*(K_{\{j\}}) \cong \tilde{H}_*(\mathbb{B}^{d-1} \cup_{f_j} \mathbb{S}^{d-2})$$

*where  $\deg(f_j) = c_j$ .*

*Proof.* Proposition 19 shows that all of the spaces  $K_A$  share the same  $(d-2)$ -skeleton as  $K_{p_1, \dots, p_d}$ , and hence they share the same homology groups  $\tilde{H}_i$  for  $i < d-2$ . Furthermore, this  $(d-2)$ -skeleton was shown in Proposition 15 to be shellable,

with top cycle/homology group  $\tilde{Z}_{d-2} \cong \mathbb{Z}^{n-\phi(n)}$ . Thus the  $i$ -dimensional homology groups with  $i < d-2$  for any  $K_A$  will vanish, in agreement with the homology of  $\mathbb{S}^{d-1}$  and  $\mathbb{S}^{d-1} \cup_{f_j} \mathbb{B}^d$ .

It only remains to show the various assertions within (i) and (ii) for  $(d-1)$ - and  $(d-2)$ -homology. Note that the complex  $K_{\{\phi(n)\}}$  is  $\mathbb{Z}$ -acyclic, by Theorem 1 and the fact that  $c_{\phi(n)} = +1$  since  $\Phi_n(x)$  is monic of degree  $\phi(n)$ . Since  $K_{\{\phi(n)\}}$  also has exactly  $n - \phi(n)$  facets  $\{F_{j \bmod n}\}_{j=\phi(n)}^{n-1}$ , their boundary cycles  $[z_{j \bmod n}]$  must form a  $\mathbb{Z}$ -basis for the  $(d-2)$ -cycle lattice  $\tilde{Z}_{d-2} \cong \mathbb{Z}^{n-\phi(n)}$ . Since the subcomplex  $K_\emptyset$  of  $K_{\{\phi(n)\}}$  has the same  $(d-2)$ -skeleton and contains all of its facets except for  $F_{\phi(n) \bmod n}$ , the assertions of (i) follow.

By assertion (i), for any  $j = 0, 1, \dots, \phi(n) - 1$ , there will be a unique integer  $c$  for which  $[z_{j \bmod n}] = c[z_{\phi(n) \bmod n}]$  in  $\tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$ . This is equivalent to the assertion that there is a  $(d-1)$ -cycle in  $K_{\{j, \phi(n)\}}$  of the form

$$[F_{j \bmod n}] - c[F_{\phi(n) \bmod n}] + \sum_{\ell=\phi(n)+1}^{n-1} b_\ell[F_{\ell \bmod n}].$$

Taking  $j' = \phi(n)$  in Theorem 2, and bearing in mind that  $c_{\phi(n)} = +1$ , this forces  $c = c_j$ , as asserted in (ii).

Lastly, note that  $K_{\{j\}}$  shares the same  $(d-2)$ -skeleton as  $K_{\{\phi(n)\}}$ , and shares most of the same facets, except for replacing the facet  $F_{\phi(n) \bmod n}$  with  $F_{j \bmod n}$ . This means that the  $(d-2)$ -boundaries of  $K_{\{j\}}$  will span the sublattice of the  $(d-2)$ -cycle lattice  $\tilde{Z}_{d-2} \cong \mathbb{Z}^{n-\phi(n)}$  in which the  $\mathbb{Z}$ -basis element  $[z_{\phi(n) \bmod n}]$  is replaced by  $[z_{j \bmod n}] = c_j[z_{\phi(n) \bmod n}]$ . Thus the  $(d-1)$ -homology of  $K_{\{j\}}$  still vanishes, and its  $(d-2)$ -homology is the quotient lattice  $\mathbb{Z}/c_j\mathbb{Z}$ . This agrees with  $\tilde{H}_i(\mathbb{B}^{d-1} \cup_{f_j} \mathbb{S}^{d-2})$  for  $i = d-1, d-2$ .  $\square$

Note that Theorem 22 does not *circumvent* Theorems 1 and Theorems 2. It uses both in a crucial way, thereby relying ultimately on the matroid duality inherent in the proofs of these results.

**Remark 23.** D. Fuchs also suggested the following further reformulation of the main results.

**Proposition 24.** *Define a  $(d-1)$ -cochain  $b$  on the complete  $d$ -partite complex  $K_{p_1, \dots, p_d}$  whose value on  $[F_{j \bmod n}]$  is  $c_j$  for  $j = 0, 1, \dots, \phi(n)$ , and 0 otherwise. Then  $b$  is a coboundary.*

*Proof.* Extend the  $\mathbb{Z}$ -basis  $\{[z_{j \bmod n}]\}_{j=\phi(n)}^{n-1}$  for the  $(d-2)$ -cycles  $\tilde{Z}_{d-2}$  to a  $\mathbb{Z}$ -basis for the  $(d-2)$ -chains  $\tilde{C}_{d-2}$  of  $K_{p_1, \dots, p_d}$ . Then let  $g$  in  $GL_{\mathbb{Z}}(\tilde{C}_{d-2})$  send this new basis to the standard basis of oriented  $(d-2)$ -faces  $[f]$ , and denote by  $f_0$  the  $(d-2)$ -face for which  $[f_0] = g[z_{\phi(n) \bmod n}]$ . Theorem 22(ii) shows that for  $j = 0, 1, \dots, \phi(n)$ , the coefficient of  $[f_0]$  when expanding  $[z_{j \bmod n}]$  in the above basis for  $\tilde{Z}_{d-2}$  is  $c_j$ . Thus  $g[z_{j \bmod n}]$  has coefficient  $c_j$  on  $[f_0]$ . This shows that the  $(d-2)$ -cochain  $[f_0]^*$  dual to  $[f_0]$  has the property that the coboundary map  $\partial^*$  sends  $g^*[f_0]^*$  to  $b$ :

$$\begin{aligned} \partial^* g^*[f_0]^*([F_{j \bmod n}]) &= [f_0]^*(g\partial[F_{j \bmod n}]) = [f_0]^*(g[z_{j \bmod n}]) \\ &= \begin{cases} c_j & \text{if } 0 \leq j \leq \phi(n) \\ 0 & \text{if } \phi(n) + 1 \leq j \leq n-1. \end{cases} \end{aligned}$$

$\square$

**Question 25.** *Is there a natural choice of a  $(d-2)$ -chain having coboundary  $b$ ?*

An affirmative answer would be helpful in writing down the coefficients of  $\Phi_n(x)$ .

We next give a homotopy-theoretic version of Theorem 22.

**Theorem 26.** *For  $d \geq 4$ , and every  $A \subseteq \{0, 1, \dots, \phi(n)\}$ , the complex  $K_A$  is simply-connected. Consequently, for  $d \neq 3$ , one has the following.*

- (i) *The complex  $K_\emptyset$  is homotopy equivalent to  $\mathbb{S}^{d-2}$ , and contains  $[z_{\phi(n) \bmod n}]$  as a fundamental  $(d-2)$ -cycle.*
- (ii) *For  $j = 0, 1, \dots, \phi(n)$ , the cyclotomic polynomial coefficient  $c_j$  gives the degree of the attaching map from the oriented boundary  $[z_{j \bmod n}]$  of the facet  $F_{j \bmod n}$  into the homotopy  $(d-2)$ -sphere  $K_\emptyset$ , with respect to the choice of  $[z_{\phi(n) \bmod n}]$  as the fundamental cycle.*
- (iii) *In particular, the complex  $K_{\{j\}}$  is homotopy equivalent to  $\mathbb{S}^{d-2} \cup_{f_j} \mathbb{B}^{d-1}$  where  $\deg(f_j) = c_j$ .*

*Proof.* For  $d = 1, 2$ , the assertions (i),(ii),(iii) follow trivially from Theorem 22.

When  $d \geq 4$ , first observe that the fundamental group of  $K_A$  is determined by its 2-skeleton, which is the same as the 2-skeleton of  $K_{p_1, \dots, p_d}$ , by Proposition 19. The latter skeleton is shellable by Proposition 15, hence homotopy equivalent to a wedge of  $(d-2)$ -spheres, and therefore simply-connected.

For the remaining assertions when  $d \geq 4$ , since  $K_\emptyset$  is simply-connected and has the homology of  $\mathbb{S}^{d-2}$  by Theorem 22(i), assertion (i) follows from a standard application of the Hurewicz isomorphism theorem [26, Theorem 5 part (ii), p.398] and the homological Whitehead theorem [12, Cor. 4.33]. Assertion (ii) now follows from Theorem 22(ii), and assertion (iii) follows combining assertions (i) and (ii).  $\square$

**Question 27.** Let  $d = 3$ , so that  $n = p_1 p_2 p_3$  for three distinct primes  $p_1, p_2, p_3$ .

- Is  $K_\emptyset$  homotopically equivalent to the circle  $\mathbb{S}^1$ ?
- Is  $K_{\{j\}}$  homotopically equivalent to  $\mathbb{B}^2 \cup_{f_j} \mathbb{S}^1$ , where  $\deg(f_j) = c_j$ , for  $j = 0, 1, \dots, \phi(n)$ ?

One might hope, for example, to achieve such homotopy equivalences by a sequence of elementary collapses. However, in the example of  $n = 105 = 3 \cdot 5 \cdot 7$ , with  $\phi(n) = 48$ , a computer exploration indicated that

- $K_\emptyset$  did not collapse down to something homeomorphic to  $\mathbb{S}^1$ ,
- the  $\mathbb{Z}$ -acyclic complexes  $K_{\{0\}}, K_{\{48\}}$  did not collapse down to a point, and
- the complex  $K_{\{7\}}$ , which Theorem 22 predicts has the homology of a real projective plane  $\mathbb{R}P^2$  since  $c_7 = -2$ , did not collapse down to an  $\mathbb{R}P^2$ .

## 8. CONCORDANCE WITH PROPERTIES OF $\Phi_n(x)$

Subtleties in the spaces  $K_{\{j\}}$  make it not yet clear whether Theorems 1 and 2 will prove useful in approaching classical questions about the coefficients of  $\Phi_n(x)$ , e.g. as discussed in [2, 10, 11, 15, 31]. Nevertheless, we briefly explain here how various well-known properties of  $\Phi_n(x)$  manifest themselves topologically in the complexes  $K_{\{j\}}$  and  $K_{\{j, j'\}}$  that appear in Theorems 1 and 2.

**8.1. Two prime factors and graphs.** When  $d = 2$  so that  $n = p_1 p_2$  is the product of only two primes, the subcomplexes  $K_{\{j\}}$  of  $K_{p_1, p_2}$  are 1-dimensional, that is, graphs. Hence their  $(d - 2)$ -dimensional homology is their 0-dimensional homology, which is always torsion-free. It follows that the only nonzero coefficients of  $\Phi_{p_1 p_2}(x)$  are  $\pm 1$ , agreeing with a well-known old observation of Migotti [22]. The explicit expansion of  $\Phi_{p_1 p_2}(x)$  is given in Elder [10], Lam and Leung [17], and Lenstra [18].

**8.2. Coefficient symmetry and simplicial automorphisms.** It is not hard to see from (1) that, for  $n > 1$ , the cyclotomic polynomial  $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$  is palindromic, that is,  $c_j = c_{\phi(n)-j}$ . For squarefree  $n = p_1 \cdots p_d$ , this coefficient symmetry is reflected in certain simplicial automorphisms of the complex  $K_{p_1, \dots, p_d}$  which we discuss here.

Note that any element of the product of symmetric groups  $\mathfrak{S}_{p_1} \times \cdots \times \mathfrak{S}_{p_d}$  that separately permutes each of the vertex sets  $K_{p_1}, \dots, K_{p_d}$  gives rise to a simplicial automorphism of  $K_{p_1, \dots, p_d}$ , with the property that it sends a positively oriented facet  $[F_{j \bmod n}]$  as in (3) to another positively oriented facet. We focus on a subgroup of these automorphisms isomorphic to the dihedral group of order  $2n$

$$D_{2n} := \langle s, r : s^2 = r^n = 1, srs = r^{-1} \rangle.$$

Let  $s, r$  act simultaneously on each vertex set  $K_{p_i}$  as follows:

$$\begin{aligned} j \bmod p_i &\xrightarrow{s} -j \bmod p_i \\ j \bmod p_i &\xrightarrow{r} j + 1 \bmod p_i. \end{aligned}$$

This induces their action on oriented facets as follows:

$$\begin{aligned} [F_{j \bmod n}] &\xrightarrow{s} [F_{-j \bmod n}] \\ [F_{j \bmod n}] &\xrightarrow{r} [F_{j+1 \bmod n}]. \end{aligned}$$

In particular, the element  $t := r^{\phi(n)} s$  will be an involution that swaps the oriented facets  $[F_{j \bmod n}], [F_{\phi(n)-j \bmod n}]$  for each  $j$ , and hence maps the complex  $K_{\{j\}}$  isomorphically onto the complex  $K_{\{\phi(n)-j\}}$ . This shows, via Theorem 1, that  $c_j = \pm c_{\phi(n)-j}$ .

Furthermore,  $t$  maps  $K_{\{j, \phi(n)\}}$  isomorphically onto  $K_{\{\phi(n)-j, 0\}}$ . This shows via Theorem 2 that  $c_j, c_{\phi(n)-j}$  must also have the same sign: their sign difference would have to be the same as the sign difference between the leading coefficient  $c_{\phi(n)} (= +1)$  and the constant coefficient  $c_0 (= +1)$ .

**8.3. Cyclotomic polynomials for even  $n$  and suspension.** It is also well-known, and follows from (1), that  $\phi(2n) = \phi(n)$  for  $n$  odd, and that the two cyclotomic polynomials

$$\Phi_{2n}(x) = \sum_{j=0}^{\phi(n)} \hat{c}_j x^j \quad \text{and} \quad \Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$$

determine each other by  $\Phi_{2n}(x) = \Phi_n(-x)$ . Equivalently,  $\hat{c}_j = (-1)^j c_j$  for all  $j$ . When  $n = p_1 \cdots p_d$  is squarefree, this manifests itself topologically as follows.

The complex  $\hat{K} := K_{2, p_1, \dots, p_d}$  relevant to  $\Phi_{2n}(x)$  is the two-point suspension  $\Sigma K$  of the complex  $K := K_{p_1, \dots, p_d}$  relevant to  $\Phi_n(x)$ . Here the two suspension vertices



in  $\hat{K}$  are labelled  $(0 \bmod 2)$  and  $(1 \bmod 2)$ . This means that every facet  $F_{\ell \bmod n}$  of the complex  $K$  is contained in exactly two facets  $\hat{F}_{\ell \bmod 2n}$  and  $\hat{F}_{\ell+n \bmod 2n}$  of  $\hat{K}$ .

Now consider the two subcomplexes  $\hat{K}_{\{j\}}, K_{\{j\}}$  of  $\hat{K}, K$  whose homology torsion predict the two coefficients  $\hat{c}_j, c_j$  up to sign. We claim that  $\hat{K}_{\{j\}}$  contains the two-point suspension  $\Sigma K_{\{j\}}$  as a deformation retract. Specifically,  $\Sigma K_{\{j\}}$  is the subcomplex of  $\hat{K}_{\{j\}}$  generated by the facets  $\{\hat{F}_{\ell \bmod 2n}\}$  as  $\ell$  runs through

$$\{j, n+j\} \cup \{\phi(n)+1, n+\phi(n)+1, \phi(n)+2, n+\phi(n)+2, \dots, n-1, 2n-1\}.$$

The collection  $\mathcal{F}$  of facets of  $\hat{K}_{\{j\}}$  lying outside the subcomplex  $\Sigma K_{\{j\}}$  is given by  $\{\hat{F}_{\ell+n \bmod 2n}\}$  for  $\ell$  in  $\{0, 1, \dots, \phi(n)\} \setminus \{j\}$ . Each such facet  $\hat{F}_{\ell+n \bmod 2n}$  in  $\mathcal{F}$  has a *free* codimension one face, namely the facet  $F_{\ell \bmod n}$  of  $K_{\{j\}}$ , because the facet  $\hat{F}_{\ell \bmod 2n}$  of  $\hat{K}$  is absent from  $\hat{K}_{\{j\}}$ . Thus one can remove the facets in  $\mathcal{F}$  from  $\hat{K}_{\{j\}}$  via a sequence of elementary collapses, leaving the retract  $\Sigma K_{\{j\}}$ . This explains why  $\hat{c}_j = \pm c_j$ .

The exact sign relationship  $\hat{c}_j = (-1)^j c_j$  comes from a similar relationship between the complexes  $\hat{K}_{\{j,j'\}}, K_{\{j,j'\}}$  of  $\hat{K}, K$ : the two-point suspension  $\Sigma K_{\{j,j'\}}$  is a deformation retract of  $\hat{K}_{\{j,j'\}}$ . One must also analyze the relation between the orientations of facets in a  $(d-2)$ -cycle  $z$  of  $K_{\{j,j'\}}$  versus the orientations of the analogous facets in the suspended  $(d-1)$ -cycle  $\Sigma(z)$  of  $\hat{K}_{\{j,j'\}}$ ; we omit this detailed analysis.

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GREGG MUSIKER, SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

*E-mail address:* `musiker@math.umn.edu`

VICTOR REINER, SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

*E-mail address:* `reiner@math.umn.edu`